

Thm: TFAE

- ① F is real closed
- ② $F(i)$ is alg. closed
- ③ $\forall a \in F$, a is a square or $-a$ is a square and every odd deg. poly has a root.

Obs: $a = b^2 \Rightarrow b^2 + c^2 = 0 \Rightarrow \frac{b^2}{c^2} = -1$
 $-a = c^2$

RCF - \mathcal{L}_R : Axioms

- a) Ordered field
- b) sum of n squares $\neq -1$
- c) $\forall n$, n is a sq. or $-n$ is a sq.
- d) every odd deg poly has a root.

Lemma: $(F, <)$ and $0 < x \in F$ not a square. Then we can extend the order to $F(\sqrt{x})$.

Pf: $0 < a + b\sqrt{x}$ iff $b = 0$ and $a > 0$ OR
 $b > 0$ and $(a > 0$ or $x > \frac{a^2}{b^2})$ OR
 $b < 0$ and $(0 > 0$ and $x < \frac{a^2}{b^2})$.

Prop: i) $(F, <)$, if has a real closure R which extends the order on F .

ii) RCF_v is the theory ordered domains.

Pf: Construct $(L, <)$ by adding square roots of the elts. in F successively.

$\{ \text{alg. real ext's of } L \}$

$(R, <)$ maximal elt. : real closed.

ii) Recall: $M \models T_v$ iff $M \hookrightarrow N$ for some $N \models T$

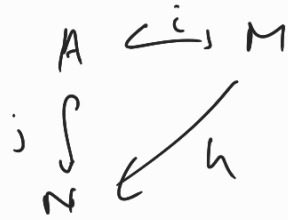
$M \models T_v$, $M \hookrightarrow (F, <) \Rightarrow M \models F$
 $\Rightarrow M$ is a domain.

$(D, <) \hookrightarrow (F, <) \hookrightarrow (R, <)$

$\frac{a}{b} > 0$, iff a, b have the same sign

$\therefore (D, <) \models \text{RCF}_v$.

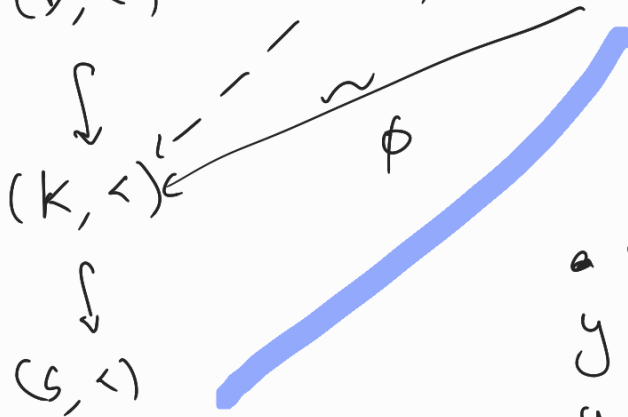
Defⁿ: T has alg. prime models iff for any $A \models T$ we have a model $M \models T$ and an $i: A \hookrightarrow M$ s.t. for all $N \models T$ and any embedding $j: A \hookrightarrow N$ we have a map $h: M \rightarrow N$ s.t. $j = hi$



Prop: $(\mathbb{F}, <)$ with R_1 and R_2 real closures which extend the order, then we have an iso $\phi: R_1 \rightarrow R_2$ s.t. $\phi|_{\mathbb{F}} = \text{id}_{\mathbb{F}}$ unique.

Th^m: RCF has alg. prime models.

pf: $(\mathbb{D}, <) \hookrightarrow (\mathbb{F}, <) \hookrightarrow (\mathbb{R}, <)$



$$K = \{ \alpha \in S : \alpha \text{ alg}/\mathbb{F} \}$$

$$\forall n > 0$$

$$y \in S, y^2 = n$$

$$y \text{ alg}/K, K \text{ alg}/\mathbb{F} \Rightarrow y \text{ alg}/\mathbb{F} \Rightarrow y \in K$$

$$p(x) \in K[x]$$

$$p(\alpha) = 0, \alpha \in S$$

$\alpha \text{ alg } K, K \text{ alg}/\mathbb{F} \Rightarrow \alpha \text{ alg}/\mathbb{F} \Rightarrow \alpha \in K$
 So, K is real closed