Thm: TFAE

1. F is real closed
2. $F(i)$ is alg. closed
3. $\forall a \in F$, a is a square or $-a$ is a square and every odd deg. poly has a root.

Obs: $a = b^2$, $-a = c^2$ $\Rightarrow b^2 + c^2 = 0 \Rightarrow \frac{b^2}{c^2} = -1$

RCF - L_{or} : Axioms

a) Ordered field
b) sum of squares $\neq -1$
c) $\forall x$, x is sq. or $-x$ is a sq.
d) every odd deg poly has a root.

Lemma: $(F, <)$ and $0 < x \in F$ not a square. Then we can extend the order to $F(\sqrt{x})$.

Pf: $0 < a + b\sqrt{x}$ iff $b = 0$ and $a > 0$ OR
$b > 0$ and $(a > 0$ or $x > \frac{a^2}{b^2})$ OR
$b < 0$ and $(a > 0$ and $x < \frac{a^2}{b^2})$.

Prop: i) $(F, <)$, it has a real closure R which extends the order on F.

ii) $RCF_\forall$ is the theory ordered domains.

Pf: Construct $(L, <)$ by adding square roots of the elts. in F successively.

$\{$alg. real ext^{n}s of $L\}$

$(R, <)$ maximal elt. : real closed.

ii) Recall: $M \models T_\forall$ iff $M \hookrightarrow N$ for some $N \models T$

$M \models T_\forall$, $M \hookrightarrow (F, <) \Rightarrow M \subseteq F$
$\Rightarrow M$ is a domain.

$(D, <) \hookrightarrow (F, <) \hookrightarrow (R, <)$

$\frac{a}{b} > 0$, iff a, b have the same sign

So, $(D, <) \models RCF_\forall$.

Def^n: T has alg. prime models iff for any $A \models T_\forall$ we have a model of $M \models T$ and an $i: A \hookrightarrow M$ s.t. for all $N \models T$ and any embedding $j: A \hookrightarrow N$ we have a map $h: M \to N$ s.t. $j = hi$

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & M \arrow[dl, "h"] \\ N & \end{tikzcd}$$

Prop: $(F,<)$ with R_1 and R_2 real closures which extend the order, then we have an unique iso $\phi: R_1 \to R_2$ s.t. $\phi|_F = id_F$

Th^m. RCF has alg. prime models.

Pf:

$$\begin{tikzcd} (D,<) \arrow[r, hook] \arrow[d, hook] & (F,<) \arrow[r, hook] \arrow[dl, dashed] & (R,<) \arrow[dll, "\phi", "\sim"'] \\ (K,<) \arrow[d, hook] & & \\ (S,<) & & \end{tikzcd}$$

$K = \{\alpha \in S : \alpha \text{ alg}/F\}$

$x > 0$

$y \in S,\ y^2 = x$.

y alg$/K$, K alg$/F \Rightarrow y$ alg$/K \Rightarrow y \in K$

$p(x) \in K[x]$

$p(\alpha) = 0,\ \alpha \in S$. α/alg K, K alg$/F \Rightarrow \alpha$ alg$/F \Rightarrow \alpha \in K$

So, K is real closed